

LEAVITT PATH ALGEBRAS OF FINITE GELFAND-KIRILLOV DIMENSION

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ABSTRACT. Groebner-Shirshov basis and Gelfand-Kirillov dimension of the Leavitt path algebra are derived.

1. INTRODUCTION.

Leavitt path algebras were introduced in [AA] as algebraic analogs of graph Cuntz-Kreiger C*-algebras. Since then they have received significant attention from algebraists. In this paper we (i) find a Groebner-Shirshov basis of a Leavitt path algebra, (ii) determine necessary and sufficient conditions for polynomially bounded growth, and (iii) find Gelfand-Kirillov dimension.

2. DEFINITIONS AND TERMINOLOGIES

A (directed) graph $\Gamma = (V, E, s, r)$ consists of two sets V and E , called vertices and edges respectively, and two maps $s, r : E \rightarrow V$. The vertices $s(e)$ and $r(e)$ are referred to as the source and the range of the edge e , respectively. The graph is called row-finite if for all vertices $v \in V$, $|(s^{-1}(v))| < \infty$. A vertex v for which $(s^{-1}(v))$ is empty is called a sink. A path $p = e_1 \dots e_n$ in a graph Γ is a sequence of edges $e_1 \dots e_n$ such that $r(e_i) = s(e_{i+1})$ for $i = 1, \dots, (n-1)$. In this case we say that the path p starts at the vertex $s(e_1)$ and ends at the vertex $r(e_n)$. If $s(e_1) = r(e_n)$, then the path is closed. If $p = e_1 \dots e_n$ is a closed path and the vertices $s(e_1), \dots, s(e_n)$ are distinct, then the subgraph $(s(e_1), \dots, s(e_n); e_1, \dots, e_n)$ of the graph Γ is called a cycle.

Let Γ be a row-finite graph and let F be a field. The Leavitt path F -algebra $L(\Gamma)$ is the F -algebra presented by the set of generators $\{v, v \in V\}$, $\{e, e^* \mid e \in E\}$ and the set of relators (1) $v_i v_j = \delta_{v_i, v_j} v_i$ for all $v_i, v_j \in V$; (2) $s(e)e = er(e) = e$, $r(e)e^* = e^*s(e) = e^*$, for all $e \in E$; (3) $e^*f = \delta_{e, fr(e)}$, for all $e, f \in E$; (4) $v = \sum_{s(e)=v} ee^*$, for an arbitrary vertex $v \in V \setminus \{\text{sinks}\}$.

The mapping which sends v to v , for $v \in V$, e to e^* and e^* to e , for $e \in E$, extends to an involution of the algebra $L(\Gamma)$. If $p = e_1 \dots e_n$ is a path, then $p^* = e_n^* \dots e_1^*$.

3. A BASIS OF $L(\Gamma)$

For an arbitrary vertex v which is not a sink, choose an edge $\gamma(v)$ such that $s(\gamma(v)) = v$. We will refer to this edge as special. In other words, we fix a function $\gamma : V \setminus \{\text{sinks}\} \rightarrow E$ such that $s(\gamma(v)) = v$ for an arbitrary $v \in V \setminus \{\text{sinks}\}$.

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Theorem 1. *The following elements form a basis of the Leavitt path algebra $L(\Gamma)$:*

(i) $v, v \in V$, (ii) p, p^* , where p is a path in Γ , (iii) pq^* , where $p = e_1 \dots e_n$, $q = f_1 \dots f_m$, $e_i, f_j \in E$, are paths that end at the same vertex $r(e_n) = r(f_m)$, with the condition that the last edges e_n and f_m are either distinct or equal, but not special.

Proof. Recall that a well-ordering on a set is a total order (that is, any two elements can be ordered) such that every non-empty subset of elements has a least element.

As a first step, we will introduce a certain well-ordering on the set of generators $X = V \cup E \cup E^*$. Choose an arbitrary well-ordering on the set of vertices V . If e, f are edges and $s(e) < s(f)$ then $e < f$. It remains to order edges that have the same source. Let v be a vertex which is not a sink. Let e_1, \dots, e_k be all the edges that originate from v . Suppose $e_k = \gamma(v)$. We order the edges as follows: $e_1 < e_2 < \dots < e_k = \gamma(v)$. Choose an arbitrary well-ordering on the set E^* . For arbitrary elements $v \in V, e \in E, f^* \in E^*$, we let $v < e < f^*$. Thus the set $X = V \cup E \cup E^*$ is well-ordered. Let X^* be the set of all words in the alphabet X . The length-lex order (see [B, Be]) makes X^* a well-ordered set. For all $v \in V$ and $e \in E$, we extend the set of relators (1) - (4) by (5): $ve = 0$, for $v \neq s(e)$; $ev = 0$, for $v \neq r(e)$; $ve^* = 0$, for $v \neq r(e)$; $e^*v = 0$, for $v \neq s(e)$. The straightforward computations show that the set of relators (1) - (5) is closed with respect to compositions (see [B, BE]). By the Composition-Diamond Lemma ([B, BE]) the set of irreducible words (not containing the leading monomials of relators (1) - (5) as subwords) is a basis of $L(\Gamma)$. This completes the proof. \square

4. LEAVITT PATH ALGEBRAS OF POLYNOMIAL GROWTH

Recall some general facts on the growth of algebras. Let A be an algebra (not necessarily unital), which is generated by a finite dimensional subspace V . Let V^k denote the span of all products $v_1 \dots v_k, v_i \in V, k \leq n$. Then $V = V^1 \subset V^2 \subset \dots, A = \cup_{n \geq 1} V^n$ and $g_{V(n)} = \dim V^n < \infty$. Given the functions f, g from $N = \{1, 2, \dots\}$ to the positive real numbers R_+ , we say that $f \preceq g$ if there exists $c \in N$ such that $f(n) \leq cg(cn)$ for all n . If $f \preceq g$ and $g \preceq f$ then the functions f, g are said to be asymptotically equivalent, and we write $f \sim g$. If W is another finite dimensional subspace that generates A , then $g_{V(n)} \sim g_{W(n)}$. If $g_{V(n)}$ is polynomially bounded, then we define the Gelfand-Kirillov dimension of A as $GKdim A = \limsup_{n \rightarrow \infty} \frac{\ln g_{V(n)}}{\ln n}$. The definition of GK -dimension does not depend on a choice of the generating space V as long as $\dim V < \infty$. If the growth of A is not polynomially bounded, then $GKdim A = \infty$.

We now focus on finitely generated algebras and we will assume that the graph Γ is finite. Let C_1, C_2 be distinct cycles such that $V(C_1) \cap V(C_2) \neq \emptyset$. Then we can renumber the vertices so that $C_1 = (v_1, \dots, v_m; e_1, \dots, e_m), C_2 = (w_1, \dots, w_n; f_1, \dots, f_n), v_1 = w_1$. Let $p = e_1 \dots e_m$, and $q = f_1 \dots f_n$.

Lemma 2. *The elements p, q generate a free subalgebra in $L(\Gamma)$.*

Proof. By Theorem 1, different paths viewed as elements of $L(\Gamma)$ are linearly independent. If u_1, u_2 are different words in two variables, then $u_1(p, q)$ and $u_2(p, q)$ are different paths. Indeed, cutting out a possible common beginning, we can assume that u_1, u_2 start with different letters, say, $u_1(p, q) = p \dots, u_2(p, q) = q \dots$. If

$m > n$ then the path $u_2(p, q)$ returns to the vertex v_1 at the n -th step, whereas $u_1(p, q)$ does not. If $m = n$, then the left segments of length m of $u_1(p, q)$, $u_2(p, q)$ are different. This proves the lemma. \square

Corollary 3. *If two distinct cycles have a common vertex, then $L(\Gamma)$ has exponential growth.*

From now on we will assume that any two distinct cycles of the graph Γ do not have a common vertex.

For two cycles C', C'' , we write $C' \implies C''$, if there exists a path that starts in C' and ends in C'' .

Lemma 4. *If C', C'' are two cycles such that $C' \implies C''$, and $C'' \implies C'$, then $C' = C''$.*

Proof. Choose a path p that starts in C' and ends in C'' . Similarly, choose a path q that starts in C'' and finishes in C' . There exists also a path p' on C'' , which connects $r(p)$ with $s(q)$ and a path q' on C' , which connects $r(q)$ with $s(p)$. Now, $pp'qq'$ is a closed path, which visits both C' and C'' . Let t be a closed path with this property (visiting both C' and C'') having a minimal length. Write $t = e_1 \cdots e_n$, $e_i \in E$. We claim that the vertices $s(e_1), \dots, s(e_n)$ are all distinct, thus $t = (s(e_1), \dots, s(e_n); e_1, \dots, e_n)$ is a cycle. Assuming the contrary, let $s(e_i) = s(e_j)$, $1 \leq i < j \leq n$, and $j - i$ is minimal with this property. Then $t' = (s(e_i), s(e_{i+1}), \dots, s(e_j); e_i, e_{i+1}, \dots, e_{j-1})$ is a cycle. Let us "cut it out", that is, consider the path $t'' = e_1 \cdots e_{i-1}e_j \cdots e_n$. This path is shorter than t . Hence t'' can not visit both C' and C'' . Suppose that t'' does not visit C' . Then at least one of the vertices $s(e_i), \dots, s(e_{j-1})$ lies in C' . Since two intersecting cycles coincide, it implies that $t' = C'$, hence $s(e_j)$ lies in C' . This contradicts our assumption that t'' does not visit C' . Hence $t = C' = C''$. This proves the lemma. \square

A sequence of distinct cycles C_1, \dots, C_k is a chain of length k if $C_1 \implies \dots \implies C_k$. The chain is said to have an exit if the cycle C_k has an exit (see [AA]), that is, if there exists an edge e such that $s(e) \in V(C_k)$, but e does not belong to C_k . Let d_1 be the maximal length of a chain of cycles in Γ , and let d_2 be the maximal length of chain of cycles with an exit. Clearly, $d_2 \leq d_1$.

Theorem 5. *Let Γ be a finite graph.*

- (1) The Leavitt path algebra $L(\Gamma)$ has polynomially bounded growth if and only if any two distinct cycles of Γ do not have a common vertex;
- (2) If d_1 is the maximal length of a chain of cycles in Γ , and d_2 is the maximal length of chain of cycles with an exit, then $GK \dim L(\Gamma) = \max(2d_1 - 1, 2d_2)$.

Proof. As in the proof of Theorem 1 we consider the generating set $X = V \cup E \cup E^*$ of $L(\Gamma)$. Let E' be the set of edges that do not belong to any cycle. Let P' be the set of all paths that are composed from edges from E' . Then an arbitrary path from P' never arrives to the same vertex twice. Hence, $|P'| < \infty$.

By Theorem 1 the space $\text{Span}(X^n)$ is spanned by elements of the following types:

- (1) a vertex,
- (2) a path $p = p'_1 p_1 p'_2 p_2 \cdots p_k p'_k$, where p_i is a path on a cycle C_i , $1 \leq i \leq k$, $C_1 \implies \dots \implies C_k$ is a chain, $p'_i \in P'$, $\text{length}(p) \leq n$,
- (3) p^* , where p is a path of the type (2),

(4) pq^* , where $p = p'_1 p_1 p'_2 p_2 \cdots p_k p'_{k+1}$, $q = q'_1 q_1 q'_2 \cdots q_s q'_{s+1}$; p_i, q_j are paths on cycles C_i, D_j respectively and $C_1 \Rightarrow \cdots \Rightarrow C_k, D_1 \Rightarrow \cdots \Rightarrow D_s$ are chains; $p'_i, q'_j \in P'$, $\text{length}(p) + \text{length}(q) \leq n$ with $r(p) = r(q)$. We will further subdivide this case into two subcases:

(4.1) $r(p) \notin V(C_k) \cup V(D_s)$;

(4.2) $r(p) \in V(C_k) \cup V(D_s)$.

We will estimate the number of products of $\text{length} \leq n$ in each of the above cases and then use the following elementary fact:

Let $(a_n)_{n \in N}$ be the sum of s sequences $(a_{in})_{n \in N}$, $1 \leq i \leq s$, $a_{in} > 0$. Then

$$\limsup_{n \rightarrow \infty} \frac{\ln a_n}{\ln n} = \max(\limsup_{n \rightarrow \infty} \frac{\ln a_{in}}{\ln n}, 1 \leq i \leq s)$$

Let us estimate the number of paths of the type (2). Fix a chain $C_1 \Rightarrow C_2 \Rightarrow \cdots \Rightarrow C_k$. If $C = (v_1, \dots, v_m; e_1, \dots, e_m)$ is a cycle, let $P_C = e_1 \dots e_m$. For a given cycle there are m such paths depending upon the choice of the starting point v_1 .

Let $|V(C_i)| = m_i$ and let P_{C_i} be any one of the m_i paths described above. Then an arbitrary path on C_i can be represented as $u' P_{C_i}^l u''$, where $\text{length}(u')$, $\text{length}(u'') \leq m_i - 1$. Hence every path of the type (2) which corresponds to the chain $C_1 \Rightarrow C_2 \Rightarrow \cdots \Rightarrow C_k$ can be represented as $p'_1 u'_1 P_{C_1}^{l_1} u''_1 \dots p'_k u'_k P_{C_k}^{l_k} u''_k p'_{k+1}$, where $p'_i \in P'_i$ and $\text{length}(u'_i), \text{length}(u''_i) \leq m_i - 1$. Clearly, $m_1 l_1 + \dots + m_k l_k \leq n$. This implies that the number of such paths $\leq n^k \leq n^{d_1}$. On the other hand, choosing a chain $C_1 \Rightarrow C_2 \Rightarrow \cdots \Rightarrow C_{d_1}$ of length d_1 , we can construct $\sim n^{d_1}$ paths of length $\leq n$. The case (3) is similar to the case (2).

Consider now the elements of length $\leq n$ of the type pq^* , $r(p) = r(q)$; the path p passes through the cycles of the chain $C_1 \Rightarrow C_2 \Rightarrow \cdots \Rightarrow C_k$ on the way, the path q passes through the cycles of the chain $D_1 \Rightarrow D_2 \Rightarrow \cdots \Rightarrow D_s$ on the way and so $p = p'_1 p_1 p'_2 \dots p_k p'_{k+1}$, where $p'_i \in P'_i$, each p_i is a path on the cycle C_i . Similarly, $q = q'_1 q_1 q'_2 \dots q_s q'_{s+1}$. Arguing as above, we see that for fixed chains $C_1 \Rightarrow C_2 \Rightarrow \cdots \Rightarrow C_k$ and $D_1 \Rightarrow D_2 \Rightarrow \cdots \Rightarrow D_s$, the number of such paths $\leq n^{k+s}$.

Suppose that the vertex $v = r(p) = r(q)$ does not lie in $V(C_k) \cup V(D_s)$. Then both cycles C_k and D_s have exits. Hence the number of paths of type (4.1) is $\leq n^{2d_2}$. On the other hand, let $C_1 \Rightarrow C_2 \Rightarrow \cdots \Rightarrow C_{d_2}$ be a chain and let e be an exit of the cycle C_{d_2} . Select paths p'_2, \dots, p'_{d_2} , where p'_i connects C_{i-1} to C_i , $p'_i \in P'_i$.

Select a path u''_1 on the cycle C_1 which connects $r(P_{C_1})$ to $s(p'_2)$, a path u'_2 in C_2 which connects $r(p'_2)$ to $s(P_{C_2})$, a path u''_2 on C_2 which connects $r(P_{C_2})$ to $s(p'_3)$, and so on. The path u''_{d_2} connects $r(P_{C_{d_2}})$ to $s(e)$.

Among the edges from $s^{-1}(s(e))$ choose a special one $\gamma(s(e))$ different from e . Then by Theorem 1, the elements

$$P_{C_1}^{l_1} u''_1 p'_2 u'_2 P_{C_2}^{l_2} u''_2 p'_3 \dots P_{C_{d_2}}^{l_{d_2}} u''_{d_2} e e^*(u''_2)^*(P_{C_{d_2}}^*)^{l_{d_2+1}} \dots (u''_1)^*(P_{C_1})^{l_{2d_2}}, l_i \geq 1, \quad (\text{A}),$$

are linearly independent. Let m be the total length of all elements other than $P_{C_i}^{l_i}$, $(P_{C_i}^*)^{l_{2d_2-i+1}}$. The number of elements in the inequality (A) above is the number of nonnegative integral solutions of the inequality

$$\sum_{i=1}^{d_2} m_i (l_i + l_{2d_2-i+1}) \leq n - m, \text{ which is } \sim n^{2d_2}.$$

Now suppose that the vertex $v = r(p) = r(q)$ lies in C_k . Assume at first that $C_k \neq D_s$. Then the chain $D_1 \Rightarrow D_2 \Rightarrow \cdots \Rightarrow D_s$ has an exit. If $k \leq s$, then the number of the paths of this type is $\leq n^{k+s} \leq n^{2d_2}$.

If $s < k$, then $n^{k+s} \leq n^{2k-1} \leq n^{2d_1} - 1$.

Next, let $C_k = D_s$. It means that the paths p'_{k+1}, q'_{s+1} are empty; p_k and q_s are both paths on the cycle C_k and in this case we have,

- (i) $p_k q_s^* = u$, if $p_k = u q_s$, is a path on C_k ,
- (ii) $p_k q_s^* = u^*$, if $q_s = u p_k$, is a path on C_k , and
- (iii) $p_k q_s^* = 0$, otherwise.

The number of such elements $p q^*$ is $\preccurlyeq n^{k+s-1} \leq n^{2d_1} - 1$.

On the other hand, let $C_1 \implies C_2 \implies \dots \implies C_{d_2}$ be a chain of cycles. Select paths $p'_2, \dots, p'_{d_1} \in P'$, p'_i connects C_{i-1} to C_i ; u'_i, u''_i are paths on the cycle C_i such that $P_{C_1} u'_1 p'_2 u'_2 P_{C_2} u''_2 p'_3 \dots P_{C_{d_1}} \neq 0$. By Theorem 1, the elements $P_{C_1}^{l_1} u'_1 p'_2 u'_2 P_{C_2}^{l_2} u''_2 p'_3 \dots u'_{d_1} P_{C_{d_1}}^{l_{d_1}} (u'_{d_1})^* (P_{C_{d_1-1}}^*)^{l_{d_1+1}} \dots (P_{C_1}^*)^{l_{2d_1-1}}$ are linearly independent provided that $l_i \geq 1, 1 \leq i \leq 2d_1 - 1$. The number of these elements is $\sim n^{2d_1-1}$. This proves Theorem 2. \square

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